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UNIFORM APPROXIMATIONS OF THE FUNDAMENTAL SOLUTION OF THE EQUATION OF INTERNAL WAVES†

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New representations of the fundamental solution of the equation of internal waves as convergent and asymptotic series are proposed using special functions. A system of three formulae is constructed, each defining a uniform approximation of the fundamental solution in some space–time domain, the union of these domains covering the whole space–time continuum. The results of a numerical experiment are presented, which show that the relative approximation error does not exceed 0.5%, while the computer time required to calculate the fundamental solution is reduced by a factor of over 200 as compared to the exact formulae. Copyright © 1996 Elsevier Science Ltd.

1. INTEGRAL REPRESENTATIONS OF THE FUNDAMENTAL SOLUTION

The linear equation of internal waves in the Boussinesq approximation has the form [1]

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+N^2\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=\frac{\partial^2 M(x,y,z,t)}{\partial z\partial t} \tag{1.1}$$

where w is a small vertical displacement of a fluid particle from the equilibrium position, M is the distribution of the mass sources, and N is the Väisälä–Brunt frequency. If $p(z)$ defines the density distribution in the equilibrium state of an incompressible fluid, then $N^2 = -g\rho'(z)/\rho(z) \geqslant 0$, and if $\varepsilon(z)$ defines the entropy distribution in the equilibrium state of an ideal gas, then $N^2 = g\varepsilon'(z)/\varepsilon(z) \geqslant 0$. We shall assume that $N = \text{const}$. Equation (1.1) holds for slow motions of a stratified gas.

The fundamental solution $\Phi(x, y, z, t)$ satisfies Eq. (1.1) with right-hand side $\delta(x)\delta(y)\delta(z)\delta(t)$.

For a historical survey and bibliography see [1, 2].

The solution of physical problems concerned with the motion of point sources, sinks, and dipoles can be expressed in terms of the derivatives of the fundamental solution with respect to the space and time variables. The solution of problems involving the motion of distributed sources or dipoles can be reduced to computing the convolution of known functions with the fundamental solution. The efficiency of a numerical solution of such problems depends on how rapidly the values of the fundamental solution can be calculated.

The function Φ can be represented in the form

$$\Phi=-\frac{1}{4\pi NR}\varphi(\lambda, Nt),\quad R^2=x^2+y^2+z^2,\quad \lambda=\frac{z}{R} \tag{1.2}$$

where the Laplace transform of φ has the form [1, 2]

$$\mathrm{L}\varphi=(p^2+N^2)^{-1/2}(p^2+N^2\lambda^2)^{-1/2} \tag{1.3}$$

(p is the dual variable to t).

Applying contour integration methods, one can change from (1.3) to the original representation [2]

$$\varphi(\lambda,\tau)=\frac{2}{\pi}\int_{|\lambda|}^{1}\frac{\sin(\tau\xi)d\xi}{\sqrt{(1-\xi^2)(\xi^2-\lambda^2)}} \tag{1.4}$$

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Because φ is an even function of λ, we can henceforth assume without loss of generality that $\lambda \geq 0$. Taking another branch of the analytic function (1.3), we can write

$$\varphi(\lambda,\tau)=\frac{2}{\pi}\int_0^{\lambda}\frac{\cos(\tau\xi)d\xi}{\sqrt{(1-\xi^2)(\lambda^2-\xi^2)}}-\frac{2}{\pi}\int_1^{\infty}\frac{\cos(\tau\xi)d\xi}{\sqrt{(\xi^2-1)(\xi^2-\lambda^2)}},\quad 0\leq\lambda<1 \tag{1.5}$$

The substitution $\operatorname{tg}^2 u = (\xi^2-\lambda^2)(1-\xi^2)$ reduces the integral (1.4) to

$$\varphi(\lambda,\tau)=\frac{2}{\pi}\int_0^{\pi/2}\frac{\sin(\tau\sqrt{1-k^2\sin^2 u})}{\sqrt{1-k^2\sin^2 u}}du,\quad k^2+\lambda^2=1 \tag{1.6}$$

One more integral representation of $\varphi(\lambda, \tau)$ can be obtained by making the substitution $2\xi = 1 + \lambda - (1-\lambda)\cos\theta$ in the integral (1.4). Then

$$\varphi_1(\lambda,\tau)=\frac{4}{\pi}\int_0^{\frac{\pi}{2}}\cos\left(\tau\frac{1-\lambda}{2}\cos\theta\right)f_1(\theta)d\theta,\ \varphi_2(\lambda,\tau)=\frac{4}{\pi}\int_0^{\frac{\pi}{2}}\sin\left(\tau\frac{1-\lambda}{2}\cos\theta\right)f_2(\theta)d\theta \tag{1.7}$$

$$\varphi(\lambda,\tau)=\varphi_1(\lambda,\tau)\sin\left(\tau\frac{1+\lambda}{2}\right)+\varphi_2(\lambda,\tau)\cos\left(\tau\frac{1+\lambda}{2}\right)$$

$$f_1(\theta)=\frac{1}{\sqrt{Q(\cos\theta)}}+\frac{1}{\sqrt{Q(-\cos\theta)}},\quad f_2(\theta)=\frac{1}{\sqrt{Q(\cos\theta)}}-\frac{1}{\sqrt{Q(-\cos\theta)}}$$

$$Q(u)=(1-\lambda)^2u^2-4(1-\lambda)(1+\lambda)u+3\lambda^2+10\lambda+3$$

The calculation of the values of φ, φ_1, φ_2 from the integral formulae (1.4)–(1.7) requires large amounts of computer time. Since the solution of the Cauchy problem and non-homogeneous problems for Eq. (1.1) is expressed in terms of various convolutions of the fundamental solution with known functions, the time required to compute the convolutions makes it difficult to solve such problems numerically. It is therefore better to obtain approximations of the fundamental solution which will make it possible to compute the values of φ with sufficient accuracy in a relatively short computer time.

2. REPRESENTATIONS OF φ AS A SERIES INVOLVING SPECIAL FUNCTIONS

We propose a number of new representations of φ. We will denote by $J_n(z)$, $P_n(z)$ and $T_n(z)$ the Bessel functions, the Legendre polynomial, and the Chebyshev polynomial [3]. The representation

$$\varphi(\lambda,\tau)=2\sum_{n=0}^{\infty}J_{2n+1}(\tau)P_n(1-2\lambda^2) \tag{2.1}$$

holds. When τ is fixed, $J_{2n+1}(\tau)$ decreases rapidly as n increases and series (2.1) converges rapidly.

To prove (2.1) we substitute the identities

$$\cos\left(\tau\sqrt{1-k^2\sin^2 u}\right)=J_0(\tau)+2\sum_{n=1}^{\infty}J_{2n}(\tau)T_{2n}(k\sin\theta)$$

$$\frac{4}{\pi}\int_0^{\frac{\pi}{2}}T_{2n}(k\sin\theta)d\theta=P_n(1-2\lambda^2)-P_{n-1}(1-2\lambda^2)$$

into (1.4), thereby obtaining

$$\frac{\partial\varphi}{\partial\tau}=2\sum_{n=0}^{\infty}J'_{2n+1}(\tau)P_n(1-2\lambda^2)$$

The functions φ_1 and φ_2 in (1.7) can also be represented by the series

$$\varphi_1 = \frac{4}{c}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1-\lambda}{c}\right)^{2n} P_{2n}\left(2\frac{1+\lambda}{2}\right)J_0^{(2n)}\left(\tau\frac{1-\lambda}{2}\right) \tag{2.2}$$

$$\varphi_2 = \frac{4}{c}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1-\lambda}{c}\right)^{2n+1} P_{2n+1}\left(2\frac{1+\lambda}{c}\right)J_0^{(2n+1)}\left(\tau\frac{1-\lambda}{2}\right)$$

$$c^2 = 3\lambda^2 + 10\lambda + 3$$

To prove (2.2) we use the identities

$$f_1(\theta) = \frac{2}{c}\sum_{n=0}^{\infty}\left(\frac{1-\lambda}{c}\cos\theta\right)^{2n} P_{2n}\left(2\frac{1+\lambda}{c}\right),$$

$$f_2(\theta) = \frac{2}{c}\sum_{n=0}^{\infty}\left(\frac{1-\lambda}{c}\cos\theta\right)^{2n+1} P_{2n+1}\left(2\frac{1+\lambda}{c}\right)$$

which are easily derived from the expression for the generating function of Legendre polynomials.

Series (2.2) converge uniformly in τ at a geometric rate. Indeed

$$|P_n(x)| \leq \left(|x| + \sqrt{|x^2-1|}\right)^n, \quad |J_0^n(x)| \leq 1$$

$$\left|\left(\frac{1-\lambda}{c}\right)^{2n} P_n\left(2\frac{1+\lambda}{c}\right)J_0^{(2n)}(x)\right| \leq \left(\frac{1-\lambda}{c}\right)^{2n}\left(2\frac{1+\lambda}{c}+\frac{1-\lambda}{c}\right)^{2n} = \left(\frac{1-\lambda}{1+3\lambda}\right)^{2n}$$

In addition, series (2.2) are uniformly asymptotic in τ as $\lambda \to 1$.

We observe that the larger the value of τ the slower the convergence of (2.1). A numerical experiment indicates that computing the values of the function from (2.1) and (2.2) does not give any substantial gain in computer time as compared to calculations using the integral formulae.

3. THE ASYMPTOTIC SERIES AS $\tau \to \infty$ AND WHEN $0 < \delta < \lambda < 1 - \delta$

We propose a new form of asymptotic series for φ as $\tau \to \infty$. Let us introduce the polynomials

$$C_n(k^2) = k^{2n}\sum_{m=0}^{n} P_m\left(\frac{1}{k^2}\right)\frac{(2n-2m-1)!!}{2^{3n-2m}}$$

$$D_n(k^2) = \lambda^{2n} C_n\left(-\frac{k^2}{\lambda^2}\right) = k^{2n}(-1)^n\sum_{m=0}^{n} P_m\left(1-\frac{1}{k^2}\right)\frac{(2n-2m-1)!!}{2^{3n-2m}}$$

We can represent φ by an asymptotic series of the form

$$\varphi \sim \frac{1}{k}\sqrt{\frac{2}{\pi\tau}}\sum_{n=0}^{\infty}\frac{(2n-1)!!}{\tau^n k^{2n}}C_n(k^2)\sin\left(\tau - \frac{\pi}{4} + \frac{n\pi}{2}\right) +$$

$$+\frac{1}{k}\sqrt{\frac{2}{\pi\lambda\tau}}\sum_{n=0}^{\infty}\frac{(2n-1)!!}{(\tau\lambda)^n k^{2n}}D_n(k^2)\sin\left(\lambda\tau + \frac{\pi}{4} + \frac{n\pi}{2}\right) \tag{3.1}$$

for $0 < \delta < \lambda < 1 - \delta$ and $\tau \to \infty$. All terms of asymptotic series (3.1) are defined apart from a function which converges to zero as $\tau \to \infty$ more rapidly than any negative power of τ.

Expansion (3.1) can be obtained by applying the stationary-phase method [4] to integral (1.6). This integral has two stationary points $u = 0$ and $u = \pi/2$. If the contribution of $u = 0$ to the asymptotic series is $I_1(k^2, \tau)$, it can easily be shown by changing the variable of integration that the contribution $I_2(k^2, \tau)$ of the point $u = \pi/2$ can be expressed in terms of $I_1(k^2, \tau)$

$$I_2(k^2,\tau)=\frac{1}{\lambda}I_1\left(-\frac{k^2}{\lambda^2},\tau\lambda\right)$$

By making the change of variable $1-\sqrt{(1-k^2\sin^2 u)} = k^2 \upsilon^2/2$ in integral (1.6) we find that $I_1(k^2, \tau)$ is equal to the contribution of the stationary point $\upsilon = 0$ to the asymptotic form of the integral

$$\frac{2}{\pi}\int_0^1 \frac{\sin(\tau(1-\tfrac{1}{2}k^2\upsilon^2))d\upsilon}{\sqrt{(1-k^2\upsilon^2/4)(1-\upsilon^2+k^4\upsilon^4/4)}}$$

Using the generating function for Legendre polynomials to represent the function multiplying the sine function as a series with even powers of υ and then estimating all the standard integrals as in the stationary-phase method [4], we obtain formulae (3.1).

Let us write down expressions for the first three polynomials C_n and D_n

$$C_0 = D_0 = 1,\quad C_1=\frac{1}{2}+\frac{k^2}{8},\quad C_2=\frac{1}{128}(48-8k^2+3k^4)$$

$$D_1=\frac{4-5k^2}{8},\quad D_2=\frac{1}{128}(48-88k^2+43k^4)$$

Substituting (3.1) into expression (1.2) for the fundamental solution, we obtain its representation as an asymptotic series. We write down the initial terms of this series and substitute the result into expression (1.2) for the fundamental solution

$$\Phi=\frac{1}{N\sqrt{32\pi}}\left(\frac{\sin(Nt-\pi/4)}{\rho\sqrt{Nt}}+\frac{\sin(Nt|z|/R+\pi/4)}{\rho\sqrt{Nt|z|/R}}-\right.$$

$$\left.-\frac{R(5\rho^2+4z^2)}{8\rho^3(Nt)^{3/2}}\sin(Nt+\pi/4)+\frac{R(4z^2-\rho^2)}{8\rho^3(Nt|z|/R)^{3/2}}\sin\left(\frac{Nt|z|}{R}-\frac{\pi}{4}\right)+\ldots\right)$$

$$\rho^2=x^2+y^2,\quad R^2=\rho^2+z^2,\quad 0<\delta<\frac{\rho}{R}<1-\delta \tag{3.2}$$

The asymptotic form (3.2) is non-uniform and is unsuitable for approximations when $z \to 0$ or $\rho \to 0$. Sections 4–6 uniform asymptotic formulae will be used to obtain an efficient numerical realization.

4. UNIFORM APPROXIMATION IN THE DOMAIN $0 \leqslant \lambda \leqslant \lambda_0 < 1$

Substituting the identities

$$\frac{1}{\sqrt{1-\xi^2}}=(k^2+\lambda^2-\xi^2)^{-1/2}=\sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!}{(2n)!!}\frac{(\lambda^2-\xi^2)^n}{k^{2n+1}},\quad 0\leqslant\xi\leqslant\lambda$$

$$\frac{1}{\sqrt{\xi^2-\lambda^2}}=\sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!}{(2n)!!}\frac{\lambda^{2n}}{k^{2n+1}}\frac{(\xi^2-1)^n}{\xi^{2n+1}},\quad \xi>1$$

into (1.5) and using the fact that

$$\frac{2}{\pi}\int_0^\lambda (\lambda^2-\xi^2)^{n-\frac{1}{2}}\cos(\tau\xi)d\xi = \frac{\lambda^n}{\tau^n}J_n(\lambda\tau)(2n-1)!!, \quad \frac{2}{\pi}\int_1^\infty \frac{(\xi^2-1)^{n-\frac{1}{2}}}{\xi^{2n+1}}\cos(\tau\xi)d\xi = \frac{a_n(\tau)}{\tau^n}$$

$$a_n(\tau) = (-1)^n\frac{2}{\pi}\int_1^{+\infty}\frac{d^n}{d\xi^n}\left(\frac{(\xi^2-1)^{n-\frac{1}{2}}}{\xi^{2n+1}}\right)\cos\left(\tau\xi+\frac{n\pi}{2}\right)d\xi \tag{4.1}$$

we obtain the following representation of $\varphi(\lambda,\tau)$ as a series

$$\varphi(\lambda,\tau) = \sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!}{(2n)!!}\frac{\lambda^n}{k^{2n+1}\tau^n}\left((2n-1)!!J_n(\lambda\tau)-\lambda^n a_n(\tau)\right) \tag{4.2}$$

$$0 \leqslant \lambda < 1-\delta < 1$$

The coefficient a_0 can be expressed in terms of an integral of the Bessel function

$$a_0(\tau) = \frac{2}{\pi}\int_1^\infty \frac{\cos(\tau\xi)d\xi}{\xi\sqrt{\xi^2-1}} = \int_\tau^\infty J_0(u)du$$

When $n \geqslant 1$ the functions $a_n(\tau)$ can no longer be represented in a simple way in terms of known special functions. We find the asymptotic form of a_1 and a_2 by taking two terms of the asymptotic series for a_1 and one term for a_2

$$a_2 = -\frac{2}{\pi}\int_1^\infty \frac{d^2}{d\xi^2}\left(\frac{(\xi^2-1)^{\frac{3}{2}}}{\xi^5}\right)\cos(\tau\xi)d\xi = -3a_0(\tau)+O(\tau^{-\frac{3}{2}})$$

$$a_1 = \frac{2}{\pi}\int_1^\infty\left(\frac{3}{\xi^4}\sqrt{1-\xi^2}-\frac{1}{\xi^2\sqrt{1-\xi^2}}\right)\sin(\tau\xi)d\xi = -J_0(\tau)+\frac{4a_0(\tau)}{\tau}+O(\tau^{-\frac{5}{2}})$$

Substituting these values into (4.2) and taking the first three terms of the series, we obtain the approximation formula

$$\varphi = \left(\frac{1}{k}-\frac{9}{8}\frac{\lambda^2}{k^5\tau^2}\right)J_0(\tau\lambda)-\left(\frac{\lambda}{2k^3\tau}-\frac{9}{4}\frac{\lambda}{k^5\tau^3}\right)J_1(\tau\lambda)-$$
$$-\left(\frac{1}{k}-\frac{2\lambda^2}{k^3\tau^2}-\frac{9}{8}\frac{\lambda^4}{k^5\tau^2}\right)\int_\tau^\infty J_0(u)du-\frac{\lambda^2}{2k^3\tau}J_0(\tau)+O(\tau^{-\frac{5}{2}}) \tag{4.3}$$
$$0 \leqslant \lambda \leqslant \lambda_0 < 1$$

The efficiency of (4.3) will be demonstrated by a numerical experiment. Formula (4.3) is more accurate the smaller the ratio λ/τ.

5. UNIFORM APPROXIMATION OF φ FOR $0 < \delta < \lambda < 1$

We use representation (1.7) and set

$$f_1(\theta) = \sum_{n=0}^{\infty}A_n(\lambda)\frac{(1-\lambda)^{2n}}{(8\lambda(1+\lambda))^{n+\frac{1}{2}}}\sin^{2n}\theta$$

$$f_2(\theta)=(1-\lambda)\cos\theta\sum_{n=0}^{\infty}B_n(\lambda)\frac{(1-\lambda)^{2n}}{(8\lambda(1+\lambda))^{n+\frac{1}{2}}}\sin^{2n}\theta \tag{5.1}$$

The method of computing the coefficients $A_n(\lambda)$ and $B_n(\lambda)$ will be demonstrated below. Substituting (5.1) into (1.7) and using the well-known integral representations of Bessel functions [4], we obtain

$$\begin{aligned}\varphi_1(\lambda,\tau)&=\sum_{n=0}^{\infty}A_n(\lambda)2^{n+1}(2n-1)!!\frac{(1-\lambda)^n J_n\left(\tau\frac{1-\lambda}{2}\right)}{\tau^n(8\lambda(1+\lambda))^{n+\frac{1}{2}}}\\ \varphi_2(\lambda,\tau)&=\sum_{n=0}^{\infty}B_n(\lambda)2^{n+1}(2n-1)!!\frac{(1-\lambda)^{n+1} J_{n+1}\left(\tau\frac{1-\lambda}{2}\right)}{\tau^n(8\lambda(1+\lambda))^{n+\frac{1}{2}}}\end{aligned} \tag{5.2}$$

The coefficients $A_n(\lambda)$ and $B_n(\lambda)$ can be determined from the equations

$$y=\frac{1}{\sqrt{Q(u)}}+\frac{1}{\sqrt{Q(-u)}}=\frac{1}{\sqrt{8\lambda(1+\lambda)}}\sum_{n=0}^{\infty}A_n(\lambda)w^n$$

$$z=\frac{1}{\sqrt{Q(u)}}-\frac{1}{\sqrt{(Q(-u)}}=\frac{(1-\lambda)u}{\sqrt{8\lambda(1+\lambda)}}\sum_{n=0}^{\infty}B_n(\lambda)w^n$$

$$w=(1-\lambda)^2(1-u^2)(8\lambda(1-\lambda))^{-1}$$

Since

$$8\lambda(1+\lambda)y^2=\frac{Q(u)+Q(-u)}{Q(u)Q(-u)}=c_0(\lambda)-c_1(\lambda)w+c_2(\lambda)w^2+\ldots=$$

$$=A_0^2(\lambda)+2A_0(\lambda)A_1(\lambda)w+(A_1^2+2A_0A_2)w^2+\ldots$$

$$c_0(\lambda)=(1+\sqrt{\lambda})^2,\quad c_1(\lambda)=1+\sqrt{\lambda}+4\lambda+\lambda^{\frac{3}{2}}+\lambda^2$$

$$c_2(\lambda)=\frac{1}{4}\left(4+3\sqrt{\lambda}+16\lambda+2\lambda^{\frac{3}{2}}+16\lambda^2+3\lambda^{\frac{5}{2}}+4\lambda^3\right)$$

equating the coefficients of the same powers of w, we obtain

$$A_0=\sqrt{c_0}=1+\sqrt{\lambda},\quad A_1=-\frac{c_1}{2A_0}=-\frac{1+\sqrt{\lambda}+4\lambda+\lambda^{\frac{3}{2}}+\lambda^2}{2(1+\sqrt{\lambda})}$$

$$A_2=\frac{c_2-A_1^2}{2(1+\sqrt{\lambda})}$$

To determine the coefficients B_n we use the identity

$$8\lambda(1+\lambda)yz=1-(1+\lambda)w+(1+\lambda+\lambda^2)w^2+\ldots=$$

$$=A_0B_0+(A_1B_0+A_0B_1)w+(A_2B_0+B_1A_1+A_0B_2)w^2+\ldots$$

Equating coefficients of like powers, we obtain

$$B_0=\frac{1}{A_0}=\frac{1}{1+\sqrt{\lambda}},$$

$$B_1 = -\frac{1+\lambda+B_0A_1}{A_0} = -\frac{1+3\sqrt{\lambda}+3\lambda^{3/2}+\lambda^2}{2(1+\sqrt{\lambda})}$$

$$B_2 = \frac{1+\lambda+\lambda^2-A_1B_1-A_2B_0}{1+\sqrt{\lambda}}$$

When $0 < \delta \leq \lambda \leq 1$ to compute φ_1 and φ_2 we will use approximations which can be obtained by taking the first three terms of series (5.2). The smaller $(1-\lambda)/\tau$ is the more accurate the approximations.

6. APPROXIMATIONS OF φ FOR SMALL τ

For small values of τ we shall use the approximation

$$\varphi(\lambda,\tau) = \frac{\sin(\lambda\tau)}{\lambda\tau} - \tau + \frac{1}{k}\int_0^{k\tau} J_0(u)du$$

$$+\lambda^2k^2\left(\frac{\tau^5}{5!} - \frac{9+3\lambda^2}{8}\frac{\tau^7}{7!} + \left(1+\lambda^4+\frac{k^2}{4}\right)\frac{\tau^9}{9!}+\ldots\right) \tag{6.1}$$

which can be obtained if $\sqrt{(1-k^2\sin^2 u)}$ in (1.6) is replaced by $\sqrt{(\lambda^2+k^2\cos^2 u)}$, the integrand is expanded in powers of τ, the integrals of even powers of $\cos u$ are computed, and terms not containing λ and not containing k are added separately. Note that taking the limit as $\lambda \to 0$ or $k \to 0$ in (6.1) gives precise results.

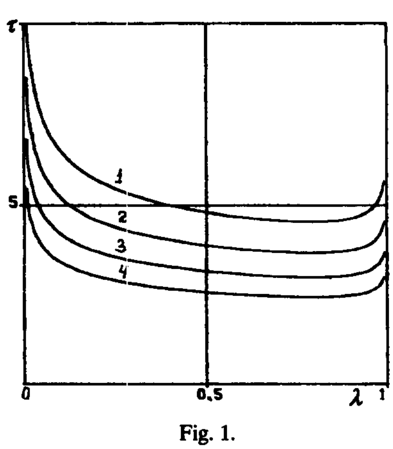

Fig. 1.

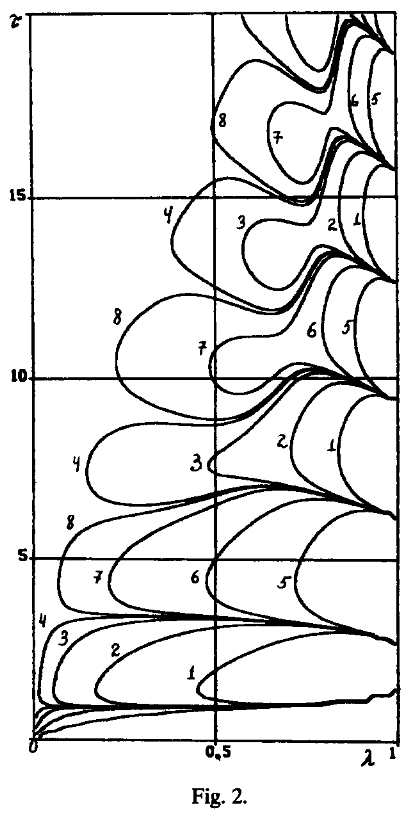

Fig. 2.

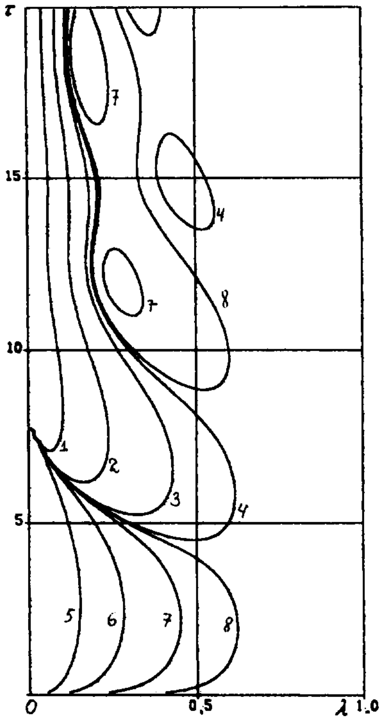

Fig. 3.

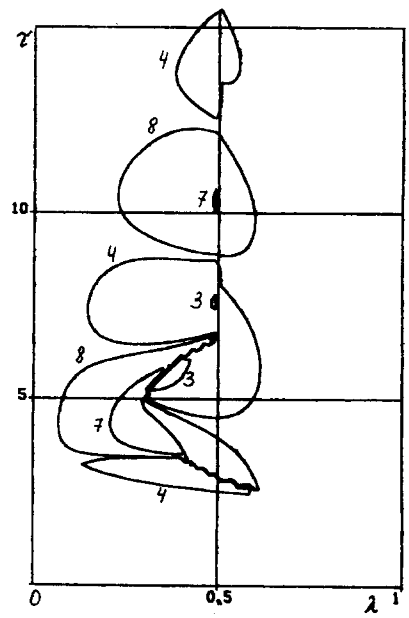

Fig. 4.

7. NUMERICAL EXPERIMENT TO VERIFY THE EFFECTIVENESS OF THE APPROXIMATION FORMULAE

The result of computing $\varphi(\lambda, \tau)$ from (1.6) will be denoted by φ. We denote by $\varphi_{\tau\to 0}$ the result obtained from the approximate formula (6.1), by $\varphi_{\lambda\to 0}$ the results from (4.3), and by $\varphi_{\lambda\to 1}$ the result from (5.2). In Figs 1–4 λ is measured along the horizontal axis and τ along the vertical axis. Computations were carried out for $0 \leqslant \tau \leqslant 20$, $0 \leqslant \lambda \leqslant 1$. In Figs 1–3 we present the level curves of the approximation errors

$$\varepsilon = \varphi - \varphi_{\tau\to 0}, \quad \varepsilon = \varphi - \varphi_{\lambda\to 0}, \quad \varepsilon = \varphi - \varphi_{\lambda\to 1}$$

The numbers 1–4 next to the level curves correspond to the errors 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}. Numbers 5–8 correspond to the negative errors -10^{-1}, -10^{-2}, -10^{-3}, -10^{-4}.

If the relative computer time required to evaluate φ is taken to be one, the time taken to compute $\varphi_{\tau\to 0}$, $\varphi_{\lambda\to 0}$, $\varphi_{\lambda\to 1}$, will be 0.0034, respectively. These times are obtained for computations on a uniform mesh for $0 \leqslant \tau \leqslant 500$, $0 \leqslant \lambda \leqslant 1$.

Let

$$a(\tau) = 0.3 + 0.05846(\tau - 4.776)^2, \quad b(\lambda)' = 4.796 - 3.739\lambda$$

Using the approximations $\varphi_{\tau\to 0}$, $\varphi_{\lambda\to 0}$, $\varphi_{\lambda\to 1}$ the global approximation φ_* can be computed for $0 \leqslant \tau < +\infty$, $0 \leqslant \lambda \leqslant 1$ as follows:

If $a(\tau) < \lambda < 0.5$, then $\varphi_* = \varphi_{\lambda\to 1}$; if $\lambda < a(\tau)$, $\tau < 3.2975$, then $\varphi_* = \varphi_{\tau\to 0}$; if $\lambda < a(\tau)$, $\tau > 3.2975$, then $\varphi_* = \varphi_{\lambda\to 1}$; if $\lambda > 0.5$, $\tau < b(\lambda)$, then $\varphi_* = \varphi_{\tau\to 0}$; if $\lambda > 0.5$, $\tau < b(\lambda)$, then $\varphi_* = \varphi_{\lambda\to 1}$.

The absolute approximation error $\varepsilon = \varphi - \varphi_*$ presented in Fig. 4 varies over the whole domain of τ, λ from -0.0028 to 0.0017, while φ itself varies from -1.04 to 1.46. The region of worst approximation lies in the vicinity of $\lambda = 0.5$, $\tau = 5$ and decreases rapidly as τ increases. The average time taken to compute φ_* is equal to 0.0045 of the time taken to compute.

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